Semisimple algebras and subfactors

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Outline

1. Semisimple algebras and subfactors
2. Small index subfactors
3. Structure theory and the Brauer group
Semisimple algebras and subfactors

Small index subfactors

Structure theory and the Brauer group
Semisimple algebras

Definition
A **semisimple algebra** over $k$ is a finite dimensional associative algebra $A$ over $k$ such that every $A$-module is semisimple.

Example
- $M_n(k)$ is a semisimple algebra over $k$.
- Direct sums of semisimple algebras are semisimple.
- If $L/K$ is a finite dimensional field extension, then $L$ is semisimple over $K$.
- The algebra of quaternions $\mathbb{H}$ is a semisimple algebra over $\mathbb{R}$.
Theorem (Artin-Wedderburn)

If $A$ is a semisimple algebra over $k$ then

$$A \cong \bigoplus_i M_{n_i}(D_i),$$

where $D_i$ is a finite dimensional division ring over $k$.

This reduces the classification of semisimple algebras over $k$ to the classification of division rings over $k$.

Example

The division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. 
The Brauer Group

Every division algebra has a center, which is some field extension of $k$.

1. Classify all field extensions $K/k$ (Galois theory).
2. Classify all division rings with center $K$.

The Brauer Group

Morita equivalence classes of central simple algebras over $K$ form a group under tensor product.

Example

The Brauer group of $\mathbb{R}$ is $\mathbb{Z}/2$ because $H \otimes_{\mathbb{R}} H \cong M_4(\mathbb{R})$. 
Subfactors

Definition

- **von Neumann algebra** is an operator algebra satisfying some properties that make it behave like a semisimple algebra. (More rigorously, it’s a $C^*$ algebra which is its own double commutant.)
- A **factor** is a von Neumann algebra with trivial center.
- A **subfactor** is an inclusion of factors $N \subset M$.
- The **Jones index** measures the “size” of the subfactor.
- We will only consider finite index subfactors of Type $II_1$ factors.

Example

If $M$ is a factor and $G$ acts outerly on $M$ then the fixed points $M^G \subset M$ is a subfactor of index $\#G$. 

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What is an algebra?

**Definition**
An algebra $A$ over $k$ is a finite dimensional vector space over $k$ together with
- unit map $k \rightarrow A$
- multiplication map $A \otimes A \rightarrow A$
which satisfies the associativity and unit axioms.

The associativity and unit axioms can be expressed as the commutativity of certain diagrams.

**Key observation**
All that we use about the category of finite dimensional $k$ vector spaces is that it has a trivial object and a tensor product. So we can talk about algebra objects in any tensor category.
Subfactors as semisimple algebras

Algebra objects from subfactors

Suppose that $N \subset M$ is a subfactor, then:
- The category of finite index $N$-$N$ bimodules is a tensor category.
- $M$ is an algebra object inside that tensor category.
- In many cases, $M$ tensor generates a fusion category.
- This is called the “standard invariant.”

Subfactors and quantum group actions

How do we understand subfactors?
- Classify unitary algebra objects in unitary tensor categories.
- For each algebra object see if it can be realized as an extension of a particular factor.
Tensor categories

**Structures**

- Abelian category
- Tensor product of representations $V \otimes W$
- Trivial representation $1$
- Associator $\omega_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$
- $u_X : 1 \otimes X \rightarrow X$
- etc.

These should satisfy some properties, for example you can associate $X \otimes Y \otimes Z \otimes W$ in different ways, you can look at $1 \otimes X \otimes Y$, etc.
The category of finite dimensional vector spaces has certain properties which make it easier to work with. We concentrate on tensor categories which satisfy these properties.

**Definition**

A *fusion category* is tensor category $\mathcal{C}$ which satisfies:

- For every object $X$ there exists a dual $X^*$.
- All Hom spaces are finite dimensional.
- $\mathcal{C}$ is semisimple.
- $\mathcal{C}$ has finitely many simple objects, up to isomorphism.
Why study fusion categories?

Fusion categories come up in many settings:

- von Neumann algebras
- Knot polynomials
- 3-dimensional TQFTs (Reshetikhin-Turaev and Turaev-Viro)
- Representation theory
- Solid state physics
- Quantum information theory
Examples of fusion categories

Trivial example
The category of vector spaces $\text{Vec}$. Only one simple object $\mathbf{1}$.

Representations of $G$
The category of representations of a finite group $\text{Rep}(G)$

$G$-graded vector spaces
The category of $G$-graded vector spaces $\text{Vec}_G$. Here the simple objects are indexed by elements of $G$ and the tensor product is given by the group structure.
One way that fusion categories get trickier is that a lot of information is encoded in the associator. For example, we can look at $G$-graded vector spaces with a nontrivial associator.

**Associator**

\[
\omega_{\alpha,\beta,\gamma} : V_{\alpha\beta\gamma} = (V_\alpha \otimes V_\beta) \otimes V_\gamma \rightarrow V_\alpha \otimes (V_\beta \otimes V_\gamma) = V_{\alpha\beta\gamma}
\]

assigns a scalar to every triple $(\alpha, \beta, \gamma)$.

- Compatibility $= \omega$ is a 3-cocycle
- $\text{Vec}(G, \omega)$ up to equivalence only depends on $\omega \in H^3(G, k^\times)$. 
Diagram categories

Where else do tensor categories come from?

Diagram categories
Objects: Boundaries of diagrams
Morphisms: (Linear comb. of) diagrams with fixed boundaries
Composition: Vertical stacking
Tensor product: Horizontal disjoint union

Temperley-Lieb
\[
\begin{array}{c}
\vdash \\
\wedge \\
\wedge \\
\end{array} + 2 \quad \begin{array}{c}
\vdash \\
\end{array} \in \{ \text{Hom}(\bullet \otimes^3 \rightarrow \bullet) \}
An example

The Fibonacci category

This fusion category has two simple objects $X$ and $1$. Furthermore, $X \otimes X \cong X \oplus 1$.

Relations

\[
\begin{align*}
\bullet + \frac{1}{\tau} & \quad = \quad \langle \rangle \\
\bullet = & \quad |
\end{align*}
\]

Where $\tau = \frac{1+\sqrt{5}}{2}$. 
Question
What does it mean for an algebra object in $\mathcal{C}$ to be semisimple?

Definition
If $A$ is an algebra in $\mathcal{C}$ a module object over $A$ is an object $M \in \mathcal{C}$ together with a map $A \otimes M \to M$ satisfying the usual conditions.

Answer
We call $A$ semisimple if the category of $A$-modules in $\mathcal{C}$ is semisimple.
Let $C = Vec_G$.

Let $A$ be the group algebra $\mathbb{C}[G]$ thought of as an object in $C$ by putting the element $g$ in the $g$-grade.

Any $A$-module is free, thus the category of $A$-modules is equivalent to $Vec$.

This algebra is simple!

Note that there’s no way to write this as a sum of smaller algebras. So semisimple algebras in other categories are interesting even when the base field is $\mathbb{C}$. 
Other examples

- $1$ is a simple algebra in a unique way.
- If $X$ is any object in a fusion category, then $X \otimes X^*$ is a semisimple algebra object with multiplication given by contraction.
- If $H$ is a subgroup of $G$, then the group ring $\mathbb{C}[H]$ is a simple algebra object in $\text{Vec}(G)$.
- The ring of functions on $G$ is a simple algebra in $\text{Rep}(G)$.
- The circle is an algebra object in the bordism category with multiplication given by the pair of pants.
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Since every object in $\text{Rep}(G)$ is a vector space, you can look at $\dim X$. But for a general fusion category the objects aren’t vector spaces.

**Example: Fibonacci category**

- $X \otimes X \cong X \oplus 1$
- $(\dim X)^2 = \dim X + 1$
- $\dim X = \frac{1 \pm \sqrt{5}}{2}$ (note that $\Box = \frac{1 + \sqrt{5}}{2}$)

**Frobenius-Perron dimension**

There’s a unique way to assign a positive real number to every object, called the Frobenius-Perron dimension.
Subfactors of small index

These correspond to small algebra objects in unitary tensor categories (called their “standard invariant”).

**Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)**

The standard invariants with index less than 4 are given by an ADE classification (via their induction/restriction graph):

$A_n, D_{2n}, E_6, E_6, E_8, E_8$

**Theorem (cf. Popa)**

The standard invariants with index exactly 4 can be classified via group theory.
In 1993 Haagerup classified possible induction/restriction graphs other than $A_\infty$ for subfactors with index less than $3 + \sqrt{3}$:
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- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- In joint work with S. Bigelow, S. Morrison, and E. Peters, we constructed the last missing case. arXiv:0909.4099
Theorem (CMS, MS, MSPP, IJMS, PT)

There are exactly ten standard invariants other than Temperley-Lieb with index between 4 and 5.

- \((\ldots, \ldots, \ldots, \ldots, \ldots\ldots, \ldots, \ldots, \ldots, \ldots)\),
- \((\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots)\),
- \((\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots)\),
- The 3\31\1 Goodman-de la Harpe-Jones standard invariant, with index \(3 + \sqrt{3}\) \((\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots)\),
- Izumi’s self-dual 2\22\1 standard invariant, with index \(\frac{5+\sqrt{21}}{2}\) \((\ldots, \ldots, \ldots, \ldots)\)

along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.
Number theory. Using TFTs you can show that $\mathbb{Q}(\dim X)$ is abelian, and hence lies in a cyclotomic extension $\mathbb{Q}(\zeta)$. This can be exploited to rule out all but finitely many possibilities in a given family. (Key contribution from Frank Calegari.)

Biunitary connections. Using Ocneanu’s theory of biunitary connections, every subfactor yields a bunch of unitary matrices satisfying some normalization conditions. Often one can rule out possibilities either just by looking at sizes of entries (and ignoring phase) or by using some properties of triangles and parallelograms to constrain the phases.

Rotational eigenvalues. Using Jones’s theory of planar algebras and rotational eigenvalues, certain numbers have to be roots of unity.
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Question

What’s the right generalization of Artin-Wedderburn, Galois theory, and the Brauer group to semisimple algebras in fusion categories?

- Again every algebra will be a direct sum of matrix algebras over analogues of division rings.
- The division rings fit into a nice structure called the Brauer-Picard groupoid (Etingof-Nikshych-Ostrik).
- The Brauer-Picard group generalizes the Brauer groups, and the arrows between different points generalizes the field extensions.
Key ideas

- Consider algebras up to **Morita equivalence**.
- If $A$ is an algebra in $C$ then $A$-mod-$A$ is another fusion category. $A$ is central simple iff $A$-mod-$A$ is equivalent to $C$ itself.

Complications

- Distinct division rings can be Morita equivalent.
- You need to account for outer automorphisms.

The Brauer-Picard groupoid (Etingof-Nikshych-Ostrik)

- A point for each fusion category of the form $A$-mod-$A$
- An arrow for each Morita equivalence class of algebras (plus choice of outer automorphism).
Example

If $\mathcal{C}$ is $\text{Vec}_R$ then we have the following Brauer-Picard groupoid:

- Points are $\text{Vec}_R$ and $\mathcal{C}$-$\text{mod}$-$\mathcal{C}$. The former has one object, the latter has two.
- There are two arrows from $\text{Vec}_R$ to itself, from $R$ and $H$.
- There are two arrows from $\mathcal{C}$-$\text{mod}$-$\mathcal{C}$ to itself coming from outer automorphisms. (Since $H^2(\text{Gal}(\mathcal{C}/R), \mathbb{C}^\times) \cong \mathbb{Z}/2$).
- There are two arrows from $\text{Vec}_R$ to $\mathcal{C}$-$\text{mod}$-$\mathcal{C}$, both corresponding to $\mathcal{C}$ but with a different choice of outer automorphism.
- There are two arrows from $\mathcal{C}$-$\text{mod}$-$\mathcal{C}$ to $\text{Vec}_R$ which correspond to $H$ and $M_2(R)$. 
Another example

What is the Brauer-Picard groupoid of $\text{Vec}(\mathbb{Z}/p)$?

- Only one point.
- Brauer-Picard group of Morita autoequivalences is the dihedral group $(\mathbb{Z}/p)^\times \rtimes \{\pm 1\}$.
- The subgroup $(\mathbb{Z}/p)^\times$ comes from outer automorphisms permuting objects.
- The other coset comes from the algebra $\mathbb{C}[\mathbb{Z}/p]$. 
Theorem (Grossman-S.)

The Brauer-Picard groupoid of the Haagerup fusion categories has three points, and only one arrow between any two points.

Applications

- One new fusion category
- Many new subfactors
- A simpler construction of the “Haagerup plus 1” subfactor.
- An interesting new “lattice of subalgebras.”
Asaeda-Haagerup

**Theorem (Grossman-S.)**

The Brauer-Picard group of the Asaeda-Haagerup fusion categories is the Klein 4 group.

**Full groupoid**

The number of points in the groupoids is more difficult. There are at least 3, but it appears that there may be 6 or 7.

**Applications**

We get at least one new fusion category (and hopefully several!), dozens of new subfactors, and interesting lattices of subalgebras. Furthermore, we hope to get a new construction of Asaeda-Haagerup.