3-dimensional topology and finite tensor categories

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Joint Meetings
Outline

1. Radford’s theorem
2. Big Picture
3. Small Picture
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Radford’s theorem for Hopf algebras

Theorem (Radford 75)

If $H$ is a finite dimensional Hopf algebra and $g \in H$ and $\alpha \in H^*$ are the distinguished grouplike elements, then:

$$S^4(x) = g(\alpha \rightarrow x \leftarrow \alpha^{-1})g^{-1}.$$

Theorem (Larson-Radford 87-88)

If $H$ is semisimple in characteristic 0, then $S^2 = 1$. 
Radford’s theorem for tensor categories

**Theorem (ENO 04)**

If $C$ is a finite tensor category and $D \in C$ is the distinguished invertible object, then there’s a canonical isomorphism of tensor functors

$$x^{****} \rightarrow D \otimes x \otimes D^{-1}.$$ 

**Theorem (ENO 04)**

If $C$ is semisimple in characteristic 0, then $D \cong 1$.

**Conjecture (ENO 02)**

If $C$ is semisimple then there’s a monoidal isomorphism $x^{**} \rightarrow x$. 
Goal of talk

Question:
Why is there a nice canonical formula for the quadruple dual? Why is the double dual more difficult?

Answer:
The double dual corresponds to the generator of $pi_1(SO_3) = \mathbb{Z}/2$

Technique:
Build a 3-dimensional fully local TFT. Joint work with Chris Douglas and Chris Schommer-Pries.
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**Topological Quantum Field Theories**

**Informal Idea**

An $n$-dimensional TQFT is an invariant of $n$-manifolds which can be computed via cutting and pasting.

**Formal Definition**

A TQFT is a symmetric monoidal functor

$$ \mathcal{F} : \text{Bord}_n \to \text{Vec}. $$

**In more detail**

- Closed $n-1$-manifolds are sent to vector spaces
- $n$-manifolds with boundaries are sent to linear maps
- Gluing goes to composition
- Disjoint union goes to tensor product
Informal Idea

It would be even better if we could cut up along lower dimensional pieces, and best of all if we could cut things up all the way down to points.

Formal Definition

An $n$-dimensional fully local topological field theory with values in a symmetric monoidal $n$-category $\mathcal{C}$ is a symmetric monoidal functor:

$$\mathcal{F} : \text{Bord}_n \to \mathcal{C}.$$
Topological structures

Flavors of TFT

TFTs come in several flavors based on what topological structures you consider:

1. unoriented
2. oriented
3. spin
4. framed (choice of trivialization of the tangent bundle)
5. etc.

Lower dimensions

In order to glue structures, we need to pick a structure on small collars of boundaries. For example, a 3-framing on a 1-manifold is a trivialization of $TM \oplus \mathbb{R}^2$. 
Dualizable objects

A symmetric monoidal $n$-category is fully dualizable if every object has a dual and ever $k$ morphism for $1 \leq k < n$ has a left adjoint and a right adjoint. Every symmetric monoidal $n$-category has a maximal fully dualizable subcategory $\mathcal{C}^{fd}$.

Theorem (Lurie(-Hopkins) ??, Baez-Dolan ??)

$$\text{TFT}^{fr}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}^{fd}$$

as spaces via

$$\mathcal{F} \mapsto \mathcal{F}(\text{pt}_+) .$$
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