The structure of the Asaeda–Haagerup subfactor

Noah Snyder (Indiana University)
joint with Izumi and Grossman

Joint math meetings
Our project (joint with Pinhas Grossman and others)

- Work in progress, also with Masaki Izumi.

Background

- Ocneanu’s “maximal atlas.”
Outline

1. Small index subfactors

Noah Snyder

The structure of the Asaeda–Haagerup subfactor
Outline

1. Small index subfactors
Suppose that $N \subset M$ is a finite index inclusion of $\mathcal{II}_1$ factors, then:

- The category of finite index $N$-$N$ bimodules is a tensor category.
- $M$ is an algebra object inside that tensor category.
- In many cases, $M$ tensor generates a fusion category.
- This is called the “standard invariant.”

How do we understand subfactors?

- Classify unitary algebra objects in unitary tensor categories.
- For each algebra object see if it can be realized as an extension of a particular factor.
Examples of fusion categories

**Trivial example**

The category of vector spaces $\text{Vec}$. Only one simple object $1$.

**Representations of $G$**

The category of representations of a finite group $\text{Rep}(G)$

**$G$-graded vector spaces**

The category of $G$-graded vector spaces $\text{Vec}_G$. Here the simple objects are indexed by elements of $G$ and the tensor product is given by the group structure.
Examples of algebra objects

- $1$ is a simple algebra in a unique way.
- If $X$ is any object in a fusion category, then $X \otimes X^*$ is a semisimple algebra object with multiplication given by contraction.
- If $H$ is a subgroup of $G$, then the group ring $\mathbb{C}[H]$ is a simple algebra object in $\text{Vec}(G)$.
- The ring of functions on $G$ is a simple algebra in $\text{Rep}(G)$.
Since every object in $\text{Rep}(G)$ is a vector space, you can look at $\dim X$. But for a general fusion category the objects aren’t vector spaces.

**Example: Fibonacci category**

- $X \otimes X \cong X \oplus 1$
- $(\dim X)^2 = \dim X + 1$
- $\dim X = \frac{1\pm\sqrt{5}}{2}$

**Frobenius-Perron dimension**

There’s a unique way to assign a positive real number to every object, called the Frobenius-Perron dimension.
Subfactors of small index

Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

The standard invariants with index less than 4 are given by an ADE classification (via their induction/restriction graph):

$A_n, D_{2n}, E_6, \overline{E_6}, E_8, \overline{E_8}$

These examples all come from quantum groups in well-understood ways.

Theorem (cf. Popa)

The standard invariants with index exactly 4 can be classified via group theory.
In 1993 Haagerup classified possible induction/restriction graphs other than $A_{\infty}$ for subfactors with index less than $3 + \sqrt{3}$:

- $\ast \leftarrow \ast \cdots$
- $\ast \cdots$
- $\ast \cdots$
- $\ast \cdots \cdots$
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Haagerup’s list

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  - $\ldots$
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The structure of the Asaeda–Haagerup subfactor
In 1993 Haagerup classified possible induction/restriction graphs other than $A_\infty$ for subfactors with index less than $3 + \sqrt{3}$:

- Haagerup and Asaeda & Haagerup (1999) constructed two of these possibilities.
- In joint work with S. Bigelow, S. Morrison, and E. Peters, we constructed the last missing case.
Theorem (CMS, MS, MSPP, IJMS, PT)

There are exactly ten standard invariants other than Temperley-Lieb with index between 4 and 5.

- \( \begin{array}{c}
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\end{array} \),
- \( \begin{array}{c}
( - - - - - - - - - - - - )
\end{array} \),
- \( \begin{array}{c}
( - - - - - - - - - - - - )
\end{array} \),
- The 3311 Goodman-de la Harpe-Jones standard invariant, with index \( 3 + \sqrt{3} \) \( \begin{array}{c}
( - - - - - - - - - - - - )
\end{array} \),
- Izumi’s self-dual 2221 standard invariant, with index \( \frac{5+\sqrt{21}}{2} \) \( \begin{array}{c}
( - - - - - - - - - - - - )
\end{array} \)

along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.
Understanding these examples

Examples coming from quantum groups

2221 and 3311 both “come from” quantum groups.

The Haagerup subfactor is increasingly well understood.

- It can be constructed in several different ways (Haagerup, Izumi, Peters)
- Izumi’s construction allows for calculations
- It appears to live in a family with $\mathbb{Z}/3\mathbb{Z}$ replaced by other groups. (Izumi & Evans-Gannon)

The remaining cases

The Asaeda–Haagerup and extended Haagerup subfactors are not as well understood.
Understanding the Asaeda–Haagerup subfactor

The Asaeda–Haagerup subfactor is in the same higher Morita equivalence class as a subfactor with index $5 + \sqrt{17}$ and the following principal graph.

- This subfactor can be constructed using Izumi’s techniques.
- The Asaeda–Haagerup subfactor can be easily reconstructed from the new subfactor.
- This allows for direct calculations. E.g. we know $Z(AH)$.
- This new subfactor may live in a family with $\mathbb{Z}/4\mathbb{Z}$ replaced by other groups.
Outline

1. Small index subfactors

The structure of the Asaeda–Haagerup subfactor
Finding all subfactors related to a specific one

Generalizing the goal

- Understand one algebra $A$ in a fusion category $\mathcal{C}$.
- Understand all algebras $A$ in a fusion category $\mathcal{C}$.
- Understand all fusion categories which appear as the category of $A$-mod-$A$ bimodules in the fusion category $\mathcal{C}$ and understand all the algebras in all of these tensor categories.
- Understand all higher Morita equivalences $\mathcal{D}M\mathcal{E}$ between fusion categories in the higher Morita equivalence class of $\mathcal{C}$.

Theorem (Ostrik)

*Any algebra object in $\mathcal{C}$ appears as the internal endomorphisms of an object in a module category $\mathcal{M}$.***
You should think of tensor categories as analogous to algebras.

<table>
<thead>
<tr>
<th>algebra</th>
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<tbody>
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<td>bimodule</td>
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Warning!

Tensor categories are higher analogs of algebras. Tensor categories have algebra objects inside them.
The Brauer-Picard groupoid of $\mathcal{C}$ (ENO)

Points
One point for each fusion category Morita equivalent to $\mathcal{C}$

Arrows
One arrow for each invertible bimodule category $\mathcal{C}M_D$.

Generalizes classical structure theory
From the BP groupoid of $\text{Vec}_k$ you can see the Brauer group of $k$, the field extensions of $k$, and the classification of all algebra over $k$.

Analogous to the Picard group
This is a higher dimensional analogue of the Picard groupoid, where points are algebras and arrows are Morita equivalences.
If $\mathcal{C}$ is $\text{Vec}_\mathbb{R}$ then we have the following Brauer-Picard groupoid:

- Points are $\text{Vec}_\mathbb{R}$ and $\mathbb{C}$-mod-$\mathbb{C}$. The former has one object, the latter has two.

- There are two arrows from $\text{Vec}_\mathbb{R}$ to itself: $\text{Vec}_\mathbb{R}$ and $\mathbb{H}$-mod.

- There are two arrows from $\mathbb{C}$-mod-$\mathbb{C}$ to itself coming from outer automorphisms. (These automorphism fix objects but have a nontrivial map $\mathcal{F}(V \otimes W) \to \mathcal{F}(V) \otimes \mathcal{F}(W)$ given by picking an element of the Galois cohomology $H^2(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \mathbb{Z}/2$.)

- There are two arrows between the two different categories, both come from $\text{Vec}_\mathbb{C}$ but one is twisted.
Algebras from the BP groupoid

- There are three distinct module categories over $\text{Vec}_R$ each with one simple object.
- Their internal endomorphisms give three division rings in $\text{Vec}_R$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$.
- There are four module categories over $\mathbb{C}\text{-mod-}\mathbb{C}$, two with one simple object and two with two simple objects.
- Their internal endomorphisms give three distinct division rings in $\mathbb{C}\text{-mod-}\mathbb{C}$ are $\mathbb{C}, M_2(\mathbb{R})$, and $\mathbb{H}$. 
Outline

1. Small index subfactors
Structure of small index subfactors

Theorem (Grossman-S. CMP)

The Brauer-Picard groupoid of the Haagerup fusion categories has three points, and only one arrow between any two points.

Applications

- One new fusion category
- Many new subfactors
- A simpler construction of the “Haagerup plus 1” subfactor.
- An interesting new lattice of intermediate subfactors.
The Brauer-Picard group of the Asaeda-Haagerup fusion categories is the Klein 4 group.

The number of points in the groupoids is exactly 6.

The center $\mathcal{Z}(\mathcal{AH})$ has 22 simple objects: one of dimension 1, six of dimension $17 + 4\sqrt{17}$, eight of dimension $32 + 8\sqrt{17}$, one of dimension $33 + 8\sqrt{17}$, and six of dimension $34 + 8\sqrt{17}$. 
Theorem (Grossman-Izumi-S. in progress)

The $T$-matrix of $\mathcal{Z}(AH)$ is the diagonal matrix:

\[
(1, 1, 1, 1, 1, -1, i, -i, -1, 1, 1, 1, e^{6i\pi/17}, e^{-6i\pi/17}, e^{10i\pi/17}, e^{-10i\pi/17}, e^{12i\pi/17}, e^{-12i\pi/17}, e^{14i\pi/17}, e^{-14i\pi/17})
\]

Theorem (Grossman-Izumi-S. in progress)

The $S$-matrix of $\mathcal{Z}(AH)$ is:

\[
\begin{pmatrix}
\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \\
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\end{pmatrix}
\]
Construct the new subfactor.

Compute the fusion rules for the dual even part and see that they agree with the fusion rules for the even part of the Asaeda–Haagerup subfactor.

Note that for one of the objects $X \otimes X \cong 1 \oplus X \oplus Y$ for a non-invertible simple object $Y$.

Prove using skein theory that $1 \oplus X$ must be an algebra.

Calculate the principal graph for $1 \oplus X$ and see that it agrees with Asaeda–Haagerup.
What makes up the BP-groupoid of AH?

**Points**
- The even parts of Asaeda–Haagerup \( \text{AH}1 \) and \( \text{AH}2 \).
- The 2-fold symmetry of \( \text{AH}2 \) yields \( \text{AH}3 \).
- The new subfactor goes between \( \text{AH}2 \) and \( \text{AH}4 \).
- The 4-fold symmetry of \( \text{AH}4 \) yields \( \text{AH}5 \) and \( \text{AH}6 \).

**Arrows**
In addition to the arrows coming from the above we have the following arrows.
- The \( \text{AH}+1 \) subfactor of index \( \frac{7 + \sqrt{17}}{2} \) between \( \text{AH}1 \) and \( \text{AH}3 \).
- The \( \text{AH}+2 \) subfactor of index \( \frac{9 + \sqrt{17}}{2} \) between \( \text{AH}1 \) and \( \text{AH}1 \).

The rest comes from (hard!) combinatorics of compositions.
Thank you!

**Our project (joint with Pinhas Grossman and others)**

- Work in progress, also with Masaki Izumi.