Progress on fusion categories

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All science is either physics or stamp collecting.

– Rutherford
Why study fusion categories?

- A fusion category is a ‘noncommutative finite group’.
- 2+1 dimensional local TFTs correspond to fusion categories (recall Chris’ talk from Monday!).
- Fusion categories are the most rigid, tractable class of higher categories.

**What is a fusion category?**

A semisimple \( \otimes \)-category with finitely many simple objects.

(Just as in Chris’ talk: abelian, enriched over Vec, monoidal, rigid.) Conjecturally all fusion categories are pivotal; let’s simplify matters by pretending this is the case. I’m also content to restrict to **unitary** fusion categories, where convenient.
Example (The ‘golden category’)

There are only 2 simple objects, called \( \iota \) and \( \tau \), with dimensions 1 and \( (1 + \sqrt{5})/2 \) and tensor products \( \tau \otimes \tau \cong \iota \oplus \tau \).

If we write the map \( \tau \otimes \tau \to \tau \) as \( \text{\includegraphics[width=1cm]{example_diagram}} \), then we can describe all morphisms as (linear combinations) of planar trivalent graphs, modulo relations

\[
\begin{align*}
\text{\includegraphics[width=2cm]{relation1}} & = \frac{1 + \sqrt{5}}{2} \\
\text{\includegraphics[width=2cm]{relation2}} & = 0
\end{align*}
\]

\[
\text{\includegraphics[width=4cm]{relation3}} = \frac{2}{1 + \sqrt{5}} \left( \text{\includegraphics[width=3cm]{relation4}} - \text{\includegraphics[width=3cm]{relation5}} \right)
\]

This is a purely diagrammatic presentation of \( \text{Rep}(U_q(\mathfrak{sl}_2)) \) at a 10-th root of unity.
What do fusion categories look like?

Fusion categories are a common generalization of (representations of) finite groups and semisimplified quantum groups at roots of unity.
But what else is out there?

We’ve discovered a number of ‘sporadic’ fusion categories, while attempting exhaustive classifications of small index subfactors.

These examples, e.g. the ‘Haagerup’, ‘Asaeda-Haagerup’, and ‘extended Haagerup’ fusion categories,

constrain conjectures on the structure of fusion categories.

**Theorem (arXiv:1002.0168)**

*The H and EH fusion categories cannot be defined over a cyclotomic field.*

These sporadic examples, in particular the Haagerup category, will appear over and over again in what follows.
What do subfactors have to do with fusion categories?

We’d like to study fusion categories up to Morita equivalence. (The 2d and 3d parts of the TQFT are the same for Morita equivalent fusion categories.)

A Morita equivalence between fusion categories $\mathcal{C}$ and $\mathcal{D}$ is an invertible bimodule category $\mathcal{C} \mathcal{M} \mathcal{D}$.

(So $\mathcal{M} \boxtimes \mathcal{M}^* \cong \mathcal{C}$ and $\mathcal{M}^* \boxtimes \mathcal{M} \cong \mathcal{D}$.)

- Every finite depth subfactor $A \subset B$ gives a Morita equivalence between the fusion categories $A \text{-- mod -- } A$ and $B \text{-- mod -- } B$.
- Conversely (in the unitary case) for every Morita equivalence and choice of generating object we obtain a subfactor.
In order to undertake classifications, we need to filter fusion categories by ‘size’. There are many interesting measures.

- rank (= # of simple objects),
- global dimension (dim(C) = \[ \sum_{V \in \text{Irr}(C)} \text{dim}(V) \]),
- generated by an object with small \( \otimes \)-multiplicities (e.g. \( X \otimes X \cong 1 \oplus X \oplus A \oplus B \)), or
- generated by an object with small dimension (or, closely related, a small index subfactor).

The rest of this talk is essentially showing you our stamp collection, organized by these measures of size.
For small rank we have

- Ostrik, “Fusion categories of rank 2” arXiv:math/0203255
- Ostrik, “Pivotal fusion categories of rank 3” arXiv:1309.4822

And in the modular case

- Bruillard-Ng-Rowell-Wang: there are finitely many modular categories in each rank. arXiv:1310.7050
- All up to rank 5 known, all weakly integral up to rank 6, all integral up to rank 7 (all pointed).
‘Near-group’ categories have a finite group $G$ worth of invertible objects, and one additional object $X$.

Izumi and Evans-Gannon \cite{arXiv:1208.1500} have studied this setting, and shown that $X^2 = nX + \sum g$, with $n = |G| - 1$ or $n = k|G|$.

- No examples with $n > |G|$ are known.
- For $n = |G| - 1$ and $n = |G|$ the fusion categories are ‘classified’ in the sense of a bijection correspondence with the (discrete) roots of some polynomials. There are many examples (indeed for ‘most’ abelian $G$), but the general pattern is not understood.

Izumi has also studied categories with a finite group $G$ of invertible objects, and then a free orbit $gX$ of other objects, and similarly classified them.
In the integer global dimension case we have:

- \( \dim(C) = p^n, pq, \) or \( pqr \) are all group theoretical (Morita equivalent to a pointed category)
  

- \( \dim(C) = p^aq^b \) are all weakly group theoretical (Morita equivalent to an iterated extension of \( \text{Vec} \) by finite groups)
  

- modular and \( \dim(C) = pq^3, pq^4, p^2q^2 \) are all group theoretical
  

- braided non-degenerate and \( \dim(C) = p^aq^bc \), with \( c \) square free, all weakly group-theoretical
  

The ambitious conjecture is that integer global dimension is equivalent to weakly group theoretical.

In the non-integer case we have essentially no results — I have a dreadfully slow program that can enumerate fusion rings up to a given global dimension, but haven’t been able to run it past 12!
Here’s a ‘low multiplicities theorem’.

**Theorem (arXiv:1302.5148)**

Let $X : A \to B$ be a 1-morphism in a nondegenerate semisimple pivotal 2-category (e.g. $X = AB_B$ for a subfactor $A \subset B$). Suppose

\[
XX^* \cong 1 \oplus Y \\
YX \cong X \oplus Z \\
ZX^* \cong Y \oplus P \oplus Q \\
PX \cong Z \oplus P' \\
QX \cong Z \oplus Q'.
\]

Then either $P'X^*$ and $Q'X^*$ have a common summand, or $X$ generates the Haagerup subfactor.

This relies on the graph planar algebra embedding theorem and an intensive Gröbner basis calculation; there may be many similar results.
Sketch.

- Let $r = \dim(P)/\dim(Q) \geq 1$.
- There is an $S \in \text{End}(X \otimes^4)$ which is a lowest weight rotational eigenvector satisfying $S^2 = (1 - r)S + rf^{(4)}$, where $f^{(4)}$ is the 4-strand Jones-Wenzl idempotent for $X$.
- If there is planar algebra satisfying the hypotheses, it embeds in its graph planar algebra — there is a linear combination $S'$ of loops of length 8 on the graph satisfying the above equations.
- If $P'X$ and $Q'X$ do not have a common summand, then all loops based at $P$ with midpoint at $Q$ stay on the known part of the principal graph.
- The subset of the equations above referring only to these loops has no solutions unless the graph norm is $\sqrt{\frac{5+\sqrt{13}}{2}}$.
- Then easily the planar algebra is Haagerup’s.
And another:

**Theorem (M-Peters-Snyder, unpublished)**

Suppose in a nondegenerate pivotal category we have $X$ simple with

$$X \otimes X \cong 1 \oplus X \oplus A \oplus B$$

and the map $X \otimes X \to X$ is rotationally invariant.

- Then if $\dim \text{Inv}(X \otimes^5) \leq 10$ then $X$ generates a $U_{qso_3}$, $U_{qg_2}$ or ‘ABA’ category.
- If $\dim \text{Inv}(X \otimes^5) = 11$ and $\dim \text{Inv}(X \otimes^6) \leq 39$ then $X$ generates an $H_3$ category (the third fusion category in the Morita equivalence class of the Haagerup subfactor).
A somewhat misleading sketch.

- Consider any two collections \( \{r_i\}_{i=1}^{40} \) and \( \{s_i\}_{i=1}^{40} \) of 40 planar trivalent graphs with 6 boundary points.
- Form the 40-by-40 matrix of inner products, with entries closed planar trivalent graphs.
- Hope we can evaluate all the entries as rational functions in a few parameters.
- Since \( \dim(\text{Inv}(X \otimes^6)) \leq 39 \), the determinant of this matrix must vanish.
- Doing this for several different collections completely determines the category.
Small index subfactors

Historically, we’ve had better classification results for small index subfactors than for fusion categories generated by a small object.

- Index has always been central to the study of subfactors.
- Ocneanu ($< 4$), Popa (4) and then Haagerup ($< 3 + \sqrt{2}$) proved the first classification results.

Translating to fusion categories, subfactor classification results give us classifications of

unitary pivotal 2-categories generated by a small 1-morphism with different source and target

(a.k.a. the representation theory of a small index subfactor, and by abuse of the Tannakian philosophy, ‘a small index subfactor’).
Every fusion category generated by an object $X$ of dimension $d$ gives a subfactor of index $d^2$.

This only remembers part of the category: $\text{Inv}((X \otimes X^*) \otimes n)$ and the planar operations thereon.
It’s worth remembering this gap. It is, however, mostly a historical accident; essentially all the techniques used in subfactor classifications could be applied to classify fusion categories generated by a small object directly. This hasn’t yet been carried out.
Theorem (Jones, Index for Subfactors)

The index of a subfactor is $4 \cos^2(\pi/n)$ or $\geq 4$.

Below 4 there are the ADE subfactors, and we have a complete classification at index 4. Besides non-amenable subfactors with trivial representation theory at every index $> 4$, there is again a discrete spectrum.

Theorem (Many, surveyed in arXiv:1304.6141)

Between index 4 and 5, we have 10 subfactors, coming in 5 pairs:

- the Haagerup subfactors
- the Asaeda-Haagerup subfactors,
- the extended Haagerup subfactors,
- the 3311 GHJ subfactors, and
- the 2221 fusion category.
Who ordered that?

- The Haagerup subfactor, which we’ve now seen three times over, is an instance of Izumi’s family of near-group categories.
- The Asaeda-Haagerup subfactor used to be mysterious, but recently Izumi-Grossman-Snyder (arXiv:1202.4396, and unpublished work) showed it is in the Morita equivalence class of an Izumi near-group category for $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.
- The extended Haagerup subfactor remains completely mysterious, apparently unrelated to anything else in the universe.
- The 3311 GHJ subfactor comes from the internal endomorphisms of the trivial object in $E_6$ as a module category over $A_{11}$.
- The 2221 category is an extension by $\mathbb{Z}/2\mathbb{Z}$ of any Izumi/Evans-Gannon near-group category for $\mathbb{Z}/3\mathbb{Z}$.
Corollary

Any object in a fusion category with dimension in \((2, \sqrt{5})\) has dimension \(\sqrt{(5 + \sqrt{21})/2}\).
Theorem (Izumi-M-Penneys-Snyder, unpublished)

At index 5, the only subfactors are group-subgroup subfactors (and the associated fusion categories are group-theoretical).


We have a classification of 1-supertransitive subfactors with index in \((5, 6\frac{1}{5}) \setminus \{6\}\).

1-supertransitive means \(X \otimes X^*\) has at least 3 summands. The examples all come from quantum groups, except for two which seem closely related (they are not braided, but satisfy similar relations), but for which we only have a brutal construction.

With Afzaly and Penneys we’re working on the general case of index in \((5, 5\frac{1}{4})\). We haven’t covered every case yet, but it appears that there are no new subfactors.
At every stage in classifications, we have encountered surprisingly (disappointingly? intriguingly!) few new examples — and these ‘special’ examples even turn up in different classifications!

[Fusion categories] are like the night sky — from down here it looks like there are huge dark patches in between a few bright stars. But every time we obtain a new telescope and zoom in on one of those dark patches, we find it’s full of stars.

— (misquoting?) Terry Gannon
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